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## CRITICAL FIELD OF A SUPERCONDUCTING FILM (II)

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# THEORY OF THE SUPERCONDUCTIVITY OF A THIN METAL FILM IN A STRONG MAGNETIC FIELD (II)*

## CRITICAL FIELD OF A SUPERCONDUCTING FILM

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ABSTRACT

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In this paper, we calculate the critical field of a superconducting thin film in the case of a second-order phase transition with the help of a theory developed in a previous paper (Ref. 1). The correction term proportional to $(\Delta t)^{3/2}$ is obtained (the magnetic field is assumed to be parallel to the surface of the film, $\Delta t = 1 - \frac{T}{T_c}$). The results obtained agree with the experimental works of Douglass-Blumberg and Toxen. /991**

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## I. INTRODUCTION

This paper is a continuation of (Ref. 1). We shall further discuss the critical field of a superconducting thin film in the case of a second order phase transition and compare the theoretical results with experiment.

A great deal of research has already been performed on the phase transition problem of a superconducting thin film in a magnetic field. It was London (Ref. 2) who first calculated the critical field of a superconducting

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** Note: numbers in the margin indicate pagination in the original foreign text.

film according to his own theory. For a very thin film ($d \ll \delta_0(T)$), the result is[1]

$$\frac{H_c}{H_{cM}(T)} = \sqrt{3}\,\frac{\delta_0(T)}{d}. \tag{1}$$

Later, Ginzburg-Landau (Ref. 3) (this will be abbreviated as GL hereafter) studied this problem in further detail, in their theory of superconductivity in a strong magnetic field. The theory of GL expects the following: The phase transition of a thin film with thickness $d < d_c$ ($d_c = \frac{\sqrt{5}}{2}\,\delta_0$) in a magnetic field is a second-order phase transition. The critical magnetic field of a second-order phase transition is

$$\frac{H_c}{H_{cM}(T)} = \sqrt{6}\,\frac{\delta_0(T)}{d}. \tag{2}$$

However, in recent years quite a few experimental results (Ref. 4-6) showed that for films which are thin enough, not only equation (1) of London's theory, but also equation (2) of the GL theory are innaccurate. For example, not long ago, Douglass-Blumberg (Ref. 6) (this will be abbreviated as DB hereafter) made extensive and accurate measurements of the critical magnetic field of a series of Sn film samples with thicknesses from $1.9 \times 10^{-6}$ cm to $4.3 \times 10^{-5}$ cm. They expressed the experimental results by the following empirical formula (in the vicinity of $T_c$):

$$H_c(\text{gauss}) = 1510\,\frac{\delta(0,d)}{2d}\,(\Delta t)^{1/2}(1 + \epsilon\Delta t), \tag{3}$$

where $\delta(0,d)$ and $\epsilon$ are coefficients independent of temperature; $\Delta t = 1 - \frac{T}{T_c}$. $\delta(0,d)$ given by DB varies considerably with d. When $d = \infty$, $\delta(0,d) = 5.1 \times 10^{-6}$ cm ( this is the generally accepted penetration depth of a large sample of SN at $T = 0°K$),but $\delta(0,d)$ increases very rapidly as d

[1] Notation used in this paper is the same as that used in (Ref. 1).

decreases. As to $\varepsilon$, DB only gave the results for some of the samples and the values lie between 0 - 0.31, and also clearly vary with d. /992

These results cannot be explained by the GL theory. Since in the GL theory, $\delta_0$ is the penetration depth of a large sample and is independent of the film thickness, the critical magnetic film of a thin film can only vary as $H_c \sim d^{-1}$. The results of DB, however, show that the variation of $H_c$ with d is faster than $d^{-1}$. Numerically, for a thin film the value of $\delta(0,d)$ measured by DB is much greater than the value of the corresponding quantity $\delta_0(0)$ in the GL theory. For example, for a thin film with thickness about $3 \times 10^{-5}$ cm, the value from the GL theory is only one quarter of that of the experiment. As for $\varepsilon$, according to the GL theory (the parameter used in the GL theory according to the formula obtained by Gor'kov), it is not difficult to prove that $\varepsilon = 0.75$. This does not agree qualitatively with experiment.

The disagreement between the theory and experiment impelled many authors (Ref. 5-3) to conduct new research on this problem. They realized that the cause of the disagreement between theory and experiment is that when the film thickness is small enough ($d < \xi_0$), even in the close vicinity of $T_c$, the relations j(r) and A(r) are non-linear. They attempted to make appropriate corrections in the London equation or GL equation, hoping to introduce a non-local effect into the local theory of London or GL. For example, Tinkham (Ref. 7) thought that in the case of a thin film the penetration depth $\delta_0(T)$ in equation (1) or equation (2) must be replaced by "the effective penetration depth of the thin film",

$$\delta(T, d) = \delta_0(T)\left(1 + \frac{\xi_0}{2d}\right) \qquad (4)$$

DB (Ref. 6) pointed out that if equation (2) is corrected in the above manner, the relation between $H_c$ and d agrees with experiment. Ittner (Ref. 5) also did a similar thing. In reference (8), Toxen applied the GL theory and expressed $H_c$ as a function of the magnetic moment. The magnetic moment is, in turn, calculated from a formula given by the non-local Pippard-Schrieffer theory (Ref. 9). Combining these two completely different theories, it is found that a result which agrees with experiment can be obtained. However, all these works have a common defect - that is, the correction made in the GL theory does not have a theoretical basis.

In reference (1), we formulated a theory for the superconductivity of thin films in a strong magnetic field and calculated the critical magnetic field of a superconducting thin film in the case of a second-order phase transition. In this paper, we have improved the calculation of the critical magnetic field in (Ref. 1) and have obtained an expression (the second section) for the critical magnetic field containing the $(\Delta t)^{3/2}$ correction term. Comparing the theoretical results with the experimental data of DB (Ref. 6) and Toxen (Ref. 7), we find the agreement very satisfactory (See third section).

## II. THE CRITICAL MAGNETIC FIELD

In Section VII of reference (1), we discussed the phase transition problem of a superconducting film in a magnetic field and reached the following conclusion. The phase transition of a superconducting film with $d < d_c$ in a magnetic field is a second-order one (Ref. 1). The critical magnetic field of a second-order phase transition is determined by the following equation:

$$K - 2d = 0, \tag{5}$$

where

$$K = |g| kT \sum_{\omega} \iint dl\, dr_1 \tilde{G}^0_{\omega}(l, r) \tilde{G}^0_{-\omega}(l, r). \tag{6}$$

$\tilde{G}^0_{\omega}(r,r')$ in equation (6) is Green's function of the normal electron in the magnetic field and is given by the expression

$$\tilde{G}^0_{\omega}(r, r') = G^0_{\omega}(r, r') + \int dl G^0_{\omega}(r, l) \frac{ie\hbar}{mc} A(l_1) \frac{\partial}{\partial l_2} G^0_{\omega}(l, r') +$$
$$+ \iint dl\, dm G^0_{\omega}(r, l) \frac{ie\hbar}{mc} A(l_1) \frac{\partial}{\partial l_2} G^0_{\omega}(l, m) \frac{ie\hbar}{mc} A(m_1) \frac{\partial}{\partial m_2} G^0_{\omega}(m, r') +$$
$$+ \iiint dl\, dm\, ds G^0_{\omega}(r, l) \frac{ie\hbar}{mc} A(l_1) \frac{\partial}{\partial l_2} G^0_{\omega}(l, m) \frac{ie\hbar}{mc} A(m_1) \frac{\partial}{\partial m_2} G^0_{\omega}(m, s) \times$$
$$\times \frac{ie\hbar}{mc} A(s_1) \frac{\partial}{\partial s_2} G^0_{\omega}(s, r') +$$

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$$+ \iiiint dl\, dm\, ds\, dz G^0_{\omega}(r, l) \frac{ie\hbar}{mc} A(l_1) \frac{\partial}{\partial l_2} G^0_{\omega}(l, m) \frac{ie\hbar}{mc} A(m_1) \frac{\partial}{\partial m_2} G^0_{\omega}(m, s) \times$$
$$\times \frac{ie\hbar}{mc} A(s_1) \frac{\partial}{\partial s_2} G^0_{\omega}(s, z) \frac{ie\hbar}{mc} A(z_1) \frac{\partial}{\partial z_2} G^0_{\omega}(z, r') +$$
$$+ \cdots.$$

According to the result of reference (1), $A(r_1) = H_0(r_1 - d) + \tilde{A}(r_1)$, and $\tilde{A}(r_1)$ is proportional to $\Delta^2$. Therefore, for the calculation of the second-order phase transition critical magnetic field, we can set $\tilde{A}(r_1) = 0$, i.e.,

$$\tilde{G}^0_{\omega}(r, r') = G^0_{\omega}(r, r') + \frac{ie\hbar H_c}{mc} \int dl G^0_{\omega}(r, l)(l_1 - d) \frac{\partial}{\partial l_2} G^0_{\omega}(l, r') +$$
$$+ \left(\frac{ie\hbar H_c}{mc}\right)^2 \iint dl\, dm G^0_{\omega}(r, l)(l_1 - d) \frac{\partial}{\partial l_2} G^0_{\omega}(l, m)(m_1 - d) \frac{\partial}{\partial m_2} G^0_{\omega}(m, r') +$$
$$+ \left(\frac{ie\hbar H_c}{mc}\right)^3 \iiint dl\, dm\, ds G^0_{\omega}(r, l)(l_1 - d) \frac{\partial}{\partial l_2} G^0_{\omega}(l, m)(m_1 - d) \frac{\partial}{\partial m_2} G^0_{\omega}(m, s) \times \tag{7}$$
$$\times (s_1 - d) \frac{\partial}{\partial s_2} G^0_{\omega}(s, r') +$$

$$+\left(\frac{ie\hbar H_C}{mc}\right)^4 \iiiint d\mathbf{l}\, d\mathbf{m}\, d\mathbf{s}\, d\mathbf{z} G^0_\omega(\mathbf{r},\mathbf{l})(l_1-d)\frac{\partial}{\partial l_2}G^0_\omega(\mathbf{l},\mathbf{m})(m_1-d)\frac{\partial}{\partial m_2}G^0_\omega(\mathbf{m},\mathbf{s})\times$$

$$\times(s_1-d)\frac{\partial}{\partial s_2}G^0_\omega(\mathbf{s},\mathbf{z})(z_1-d)\frac{\partial}{\partial z_2}G^0_\omega(\mathbf{z},\mathbf{r}')+ \qquad \text{(7 Cont'd)}$$

$$+\cdots.$$

In (Ref. 1), we substitute (7) into (6), neglect terms containing $H_c^4$ and higher powers of $H_c$, and obtain the second-order phase transition critical magnetic field as follows:

$$\frac{H_C^2}{H_{CM}^2}=3\frac{\delta_0^2(T)}{d^2}\frac{\xi_0}{d}\frac{1}{\Phi_1(\sigma)}, \tag{8}$$

where $\Phi_1(\sigma)$ is given by equation (39a) in (Ref. 1). In obtaining this result, we replace the temperature factors appearing in the equation, T and $\ln\frac{T_c}{T}$, respectively, by $T_c$ and $\Delta t$. $H_c$ is given by (8) and is proportional to $(\Delta t)^{1/2}$.

In order to explain the experiment of DB, we have to find the expression of the critical magnetic field which is accurate to $(\Delta t)^{3/2}$. Here we need not only more accurate approximations for the temperature factors T and $\ln\frac{T_c}{T}$, but also we must keep the $H_c^4$ term in the expansion of K with respect to $H_c$. K can be expressed as, which is accurate to the $H_c^4$ term,

$$K=K_0+K_2+K_4, \tag{9}$$

where

$$K_0=|g|kT\sum\iint d\mathbf{l}\, dr_1 G^0_\omega(\mathbf{l},\mathbf{r})G^0_{-\omega}(\mathbf{l},\mathbf{r}),$$

$$K_2=2\left(\frac{ie\hbar H_C}{mc}\right)^2|g|kT\sum_\omega\int\cdots\int d\mathbf{l}\, d\mathbf{s}\, d\mathbf{m}\, dr_1 G^0_\omega(\mathbf{l},\mathbf{r})G^0_{-\omega}(\mathbf{l},\mathbf{s})(s_1-d)\times$$

$$\times\frac{\partial}{\partial s_2}G^0_{-\omega}(\mathbf{s},\mathbf{m})(m_1-d)\frac{\partial}{\partial m_2}G^0_{-\omega}(\mathbf{m},\mathbf{r})+$$

$$+\left(\frac{ie\hbar H_C}{mc}\right)^2|g|kT\sum_\omega\int\cdots\int d\mathbf{l}\, d\mathbf{s}\, d\mathbf{m}\, dr_1 G^0_\omega(\mathbf{l},\mathbf{s})(s_1-d)\times$$

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$$\times\frac{\partial}{\partial s_2}G^0_\omega(\mathbf{s},\mathbf{r})G^0_{-\omega}(\mathbf{l},\mathbf{m})(m_1-d)\frac{\partial}{\partial m_2}G^0_{-\omega}(\mathbf{m},\mathbf{r}),$$

$$
\begin{aligned}
K_4 = 2\left(\frac{ie\hbar H_c}{mc}\right)^4 |g|kT \sum_{\omega} \int \cdots \int d\mathbf{l}\, d\mathbf{m}\, d\mathbf{s}\, d\mathbf{z}\, d\boldsymbol{\lambda}\, d\mathbf{r}_1 G^0_{\omega}(\mathbf{l}, \mathbf{s})(s_1 - d) \times \\
\times \frac{\partial}{\partial s_2} G^0_{\omega}(\mathbf{s}, \mathbf{r}) G^0_{-\omega}(\mathbf{l}, \mathbf{m})(m_1 - d) \frac{\partial}{\partial m_2} G^0_{-\omega}(\mathbf{m}, \mathbf{z})(z_1 - d) \times \\
\times \frac{\partial}{\partial z_2} G^0_{-\omega}(\mathbf{z}, \boldsymbol{\lambda}) (\lambda_1 - d) \frac{\partial}{\partial \lambda_2} G^0_{-\omega}(\boldsymbol{\lambda}, \mathbf{r}) + \\
+ 2\left(\frac{ie\hbar H_c}{mc}\right)^4 |g|kT \sum_{\omega} \int \cdots \int d\mathbf{l}\, d\mathbf{m}\, d\mathbf{s}\, d\mathbf{z}\, d\boldsymbol{\lambda}\, d\mathbf{r}_1 G^0_{\omega}(\mathbf{l}, \mathbf{s})(s_1 - d) \times \\
\times \frac{\partial}{\partial s_2} G^0_{\omega}(\mathbf{s}, \mathbf{r}) G^0_{-\omega}(\mathbf{l}, \mathbf{m})(m_1 - d) \frac{\partial}{\partial m_2} G^0_{-\omega}(\mathbf{m}, \mathbf{z})(z_1 - d) \times \\
\times \frac{\partial}{\partial z_2} G^0_{-\omega}(\mathbf{z}, \boldsymbol{\lambda}) (\lambda_1 - d) \frac{\partial}{\partial \lambda_2} G^0_{-\omega}(\boldsymbol{\lambda}, \mathbf{r}) + \\
+ 2\left(\frac{ie\hbar H_c}{mc}\right)^4 |g|kT \sum_{\omega} \int \cdots \int d\mathbf{l}\, d\mathbf{m}\, d\mathbf{s}\, d\mathbf{z}\, d\boldsymbol{\lambda}\, d\mathbf{r}_1 G^0_{\omega}(\mathbf{l}, \mathbf{s})(s_1 - d) \times \\
\times \frac{\partial}{\partial s_2} G^0_{\omega}(\mathbf{s}, \mathbf{m}) (m_1 - d) \frac{\partial}{\partial m_2} G^0_{\omega}(\mathbf{m}, \mathbf{r}) \times \\
\times G^0_{-\omega}(\mathbf{l}, \mathbf{z})(z_1 - d) \frac{\partial}{\partial z_2} G^0_{-\omega}(\mathbf{z}, \boldsymbol{\lambda}) (\lambda_1 - d) \frac{\partial}{\partial \lambda_2} G^0_{-\omega}(\boldsymbol{\lambda}, \mathbf{r}).
\end{aligned}
$$

In equation (9), terms containing odd powers of $H_c$ are zero, and have been discarded. $K_0$ and $K_2$ are calculated in (Ref. 1):

$$K_0 = 2d\left[|g|N(0)\ln\frac{T_c}{T} + 1\right], \tag{10}$$

$$K_2 = -|g|N(0)\frac{31\zeta(5)}{2\pi\gamma}\frac{e^2\xi_0 H_0^2 d^4}{\hbar^2 c^2}\frac{T_c}{T}\Phi_1(\sigma), \tag{11}$$

where $\gamma$ = ln C; C is the Euler constant which is 0.5772. For purposes of convenience, the expression for $\Phi_1(\sigma)$ is rewritten as:

$$
\begin{aligned}
\Phi_1(\sigma) &= \frac{32}{31\zeta(5)}\sum_{l=0}^{\infty}\frac{\Phi_0\left(\frac{\sigma}{2l+1}\right)}{(2l+1)^5}, \\
\Phi_0\left(\frac{\sigma}{2l+1}\right) &= \frac{16}{\pi^3}\left[G\left(\frac{\sigma}{2l+1}\right) - \frac{\sigma}{2l+1}F\left(\frac{\sigma}{2l+1}\right)\right], \\
G(\eta) &= \sum^{\infty} \frac{1}{\cdots}\,\mathrm{tg}^{-1}\frac{1}{\cdots},
\end{aligned}
\tag{12}
$$

$$F(\eta) = \sum_{n=0}^{\infty} \frac{1}{2n+1}\left[1 - (2n+1)\eta \,\mathrm{tg}^{-1} \frac{1}{(2n+1)\eta}\right],$$ /995

$$\sigma = 0.364 \frac{2d}{\xi_0} \frac{T}{T_c},$$

$$\xi_0 = \frac{\gamma}{\pi^2} \frac{\hbar v_0}{kT_c} \approx 0.182 \frac{\hbar v_0}{kT_c}. \qquad \text{(12 Cont'd)}$$

Functions $F(\eta)$ and $G(\eta)$, when $\eta < 1$, can be expressed as (See the Appendix)

$$F(\eta) = \frac{1}{2} \ln \frac{1}{\eta} + \frac{1}{2}(C + \ln 2 - 1) + \frac{\eta^2}{12} + O(\eta^4),$$

$$G(\eta) = \frac{\pi^3}{16} - \frac{1}{2} \eta \ln \frac{1}{\eta} - \frac{1}{2}(C + \ln 2 + 1)\eta + \frac{1}{36}\eta^3 + O(\eta^5). \qquad (13)$$

Therefore, when $\sigma < 1$, $\Phi_1(\sigma)$ can be approximately expressed as

$$\Phi_1(\sigma) = \frac{32}{31\zeta(5)} \left\{\sum_{l=0}^{\infty} \frac{1}{(2l+1)^5} - \frac{16\sigma}{\pi^3} \sum_{l=0}^{\infty} \frac{1}{(2l+1)^6}\left[\ln \frac{2l+1}{\sigma} + (C + \ln 2) + + \cdots\right]\right\}.$$

The first summation inside the braces can easily be calculated. The second sum itself makes a smaller contribution than the first one and converges very rapidly; therefore, it is sufficient to consider the first term only. Therefore we obtain, for $\sigma < 1$,

$$K_2 = -|g|N(0) \frac{31\zeta(5)e^2\xi_0 H_0^2 d^4}{2\pi\gamma\hbar^2 c^2}\left(\frac{T_c}{T}\right)\left\{1 - 0.185 \frac{2d}{\xi_0}\left[2.29 + \ln \frac{\xi_0}{2d} + \ln \frac{T_c}{T} + + \cdots\right]\right\}. \qquad (14)$$

In the vicinity of $T_c$, expanding with respect to $\Delta t = 1 - \frac{T}{T_c}$ to the first order of $\Delta t$, $K_2$ can be expressed as

$$K_2 = -|g|N(0) \frac{31\zeta(5)\, e^2\xi_0 H_0^2 d^4}{2\pi\gamma \quad \hbar^2 c^2}[1 - \mu(x)][1 + [1 + \epsilon_1(x)]\Delta t], \qquad (15)$$

where

$$\left.\begin{aligned} \mu(x) &= 0.185\, x\left(2.29 + \ln \frac{1}{x} + \cdots\right), \\ \epsilon(x) &= \frac{0.185\, x\left(1.29 + \ln \frac{1}{x} + \cdots\right)}{[1 - \mu(x)]}, \\ x &= \frac{2d}{\xi_0}. \end{aligned}\right\} \qquad (16)$$

Expanding $\ln \frac{T_c}{T}$ with respect to $\Delta t$ to the second order of $\Delta t$, $K_0$ can be written as

$$K_0 = 2d\left\{1 + |g|N(0)\Delta t\left(1 + \frac{\Delta t}{2}\right)\right\}. \tag{17}$$

As to the integration $K_4$, it can be obtained by a method similar to the one used in (Ref. 1) to calculate $K_2$. After somewhat complicated calculations we obtain, for $\sigma \ll 1$,

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$$K_4 = |g|N(0)\frac{B}{\pi\gamma^3}\left(\frac{e}{\hbar c}\right)^4 \xi_0^3 H_0^4 d^6 \left(\frac{T_c}{T}\right)^3. \tag{18}$$

where B is a constant and is calculated as $B \approx 12.5$.

Substituting (15)-(18) into (9) and (5) we obtain an equation for determining the critical magnetic field $H_c$ of a thin film in the case of second-order phase transition:

$$a_1 + a_2 H_C^2 + a_3 H_C^4 = 0, \tag{19}$$

where

$$\begin{aligned} a_1 &= \Delta t\left(1 + \frac{\Delta t}{2}\right), \\ a_2 &= -\frac{31\zeta(5)}{4\pi\gamma}\frac{e^2\xi_0 d^3}{\hbar^2 c^2}[1 - \mu(x)][1 + [1 + \epsilon_1(x)]\Delta t], \\ a_3 &= \frac{B}{2\pi\gamma^3}\left(\frac{e}{\hbar c}\right)^4 \xi_0 d^5. \end{aligned} \tag{20}$$

From (19) $H_c$ can be solved immediately:

$$H_C = \left(\frac{\hbar c}{e}\right)\frac{2.36}{\xi_0^{1/2} d^{3/2}[1 - \mu(x)]^{1/2}}(\Delta t)^{1/2}(1 + \epsilon\Delta t), \tag{21}$$

where

$$\begin{aligned} \epsilon &= \epsilon_2 - \frac{\epsilon_1}{2} - 0.25, \\ \epsilon_2 &= \frac{0.0136B}{x[1 - \mu(x)]^2} \approx \frac{0.163}{x[1 - \mu(x)]^2}. \end{aligned} \tag{22}$$

See equation (16) for the definitions of $\epsilon_1$ and $\mu(x)$. We should note that when $\sigma < 1$, $\Phi(\sigma) = 1 - \mu\left(\frac{2d}{\xi_0}\right)$. It is easy to see that the $(\Delta t)^{1/2}$-order term in equation (21) agrees with the $H_c$ given in (Ref. 1). (See formula (8) of this paper.)

## III. COMPARISON OF THEORY AND EXPERIMENT

In the above section we found the critical magnetic field (21) of a superconducting thin film at a temperature near $T_c$ for second-order phase transition. This expression is applicable, at least to films satisfying the conditions for the thickness

$$\sigma = 0.364 \frac{2d}{\xi_0} < 1 \tag{23a}$$

and

$$0.36 p_0(2d) \frac{2d}{\xi_0} \gg 1 \tag{23b}$$

(Condition (23b) is discussed in (Ref. 1)). With Sn, for $\xi_0 \sim 2.3 \times 10^{-5}$ cm, $v_0 \sim 0.65 \times 10^8$ cm/sec (Ref. 11). It is estimated from (23) that: $10^{-6}$ cm $<< 2d < 6.3 \times 10^{-5}$ cm$^{(1)*}$. In the experiment of DB, the thickness of Sn films ranged from $1.9 \times 10^{-5}$ cm to $4.3 \times 10^{-5}$ cm, and falls within the range where equation (21) is applicable.

In order to compare with the DB experiment, we express (21) in the form of (3). Obviously, the parameter $\delta(0,d)$ in (3) and the theoretical expression of $\varepsilon$ are given by

*Note: Illegible in the original foreign text.

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$$\delta(0, d) = \frac{2.36\hbar c}{1510 e \xi_0^{1/2} (2d)^{1/2} [1 - \mu(x)]^{1/2}} \quad (1) \tag{24}$$

and (22), respectively. The experimental points of DB and the $\delta(0,d)$ curve calculated according to (24) at $\xi_0 = 2.0 \times 10^{-5}$ cm are shown in Figure 1. For purposes of comparison, the result (2) from the GL theory is also drawn, using the relations $H_{cM} = 1.74\, H_{cM}(0)\, \Delta t$ and $\delta_0 = \delta_L(0)/\sqrt{2\Delta t}$ and choosing the values of $\delta\,(0,d)$ obtained at $H_{cM} = 307$ gauss (Ref. 12) and $\delta_L(0) = 3.55 \times 10^{-6}$ cm. It is a horizontal line. It can be seen that the result derived in our paper agrees very well with experiment. We should note that we are comparing the purely theoretical values with the absolute experimental values and not the scaled values. The $\xi_0$ value we used, $\xi_0 = 2.0 \times 10^{-5}$ cm, agrees reasonably well with $\xi_0 = 2.3 \times 10^{-5}$ cm given by reference (11) and $\xi_0 = 2.1 \times 10^{-5}$ cm (Ref. 13) given by Faber-Pippard.

We would like to point out that the factor $\Phi_1^{1/2}(\sigma) = (1 - \mu(x))^{1/2}$ in (24) plays an important role in making the theory agree with experiment. In order to explain this point, we plotted by means of a broken line (b), the $\delta(0,d) \sim d$ curve (taking $\xi_0 = 2.0 \times 10^{-5}$ cm) calculated according to

$$\delta(0, d) = \frac{2.36\hbar c}{1510 e \xi_0^{1/2} (2d)^{1/2}} \tag{24a}$$

This agrees with the experiment only when d is very small. However when one

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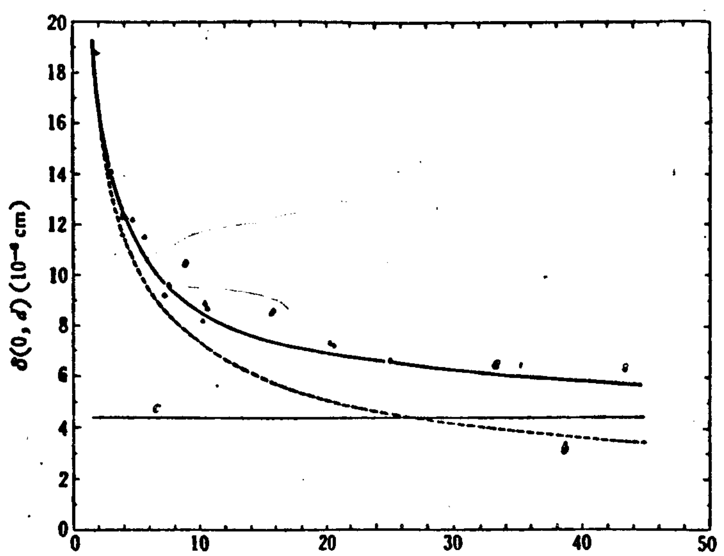
(1) More generally, from equation (8), we obtain

$$\delta(0, d) = \frac{2.36\hbar c}{1510 e \xi_0^{1/2} (2d)^{1/2}} \frac{1}{\Phi_1(\sigma)},$$

This formula is valid for all films with 2d >> 2d*. Calculation shows that it agrees with (24) within the range of film thickness used in the DB experiment.

adjusts the value of $\xi_0$, no agreement with experiment can be obtained. From this, it can be seen that the formula for the critical field of a thin film ($H_c \sim d^{-3/2}$) corresponding to (24a) is valid only when the film thickness is very small.

The theoretical curve of the coefficient $\varepsilon$ of the $(\Delta t)^{-3/2}$ term of the critical magnetic field, $H_c$ [according to formula (22) and also taking $\xi_0 = 2.0 \times 10^{-5}$ cm] and the experimental data of DB are shown in Figure 2. It can be seen that the $\varepsilon$ value given by this paper varies considerably with d.

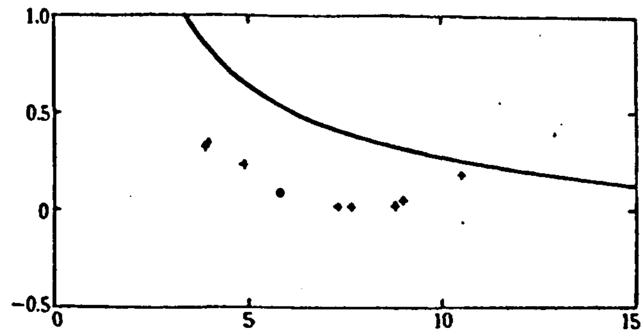



Thickness 2d ($10^{-6}$ cm)

Figure 1

Comparison of the Theoretical Values and Experimental Data of DB of $\delta(0,d)$ for Sn Films; "+" Represents DB Experimental Points. /998

As d decreases, $\varepsilon$ increases rapidly. These characteristics agree with the DB experiment in their qualitative aspect. However, quantitatively speaking, the $\varepsilon$ values given by this paper are larger than the DB experimental




Thickness 2 d ($10^{-6}$ cm)

Figure 2

Variation of ε Values with the Thickness 2d for Sn Film; "+" Represents DB Experimental Points.

values.

The $(\Delta t)^{3/2}$ term of the critical magnetic field is very small, and its contribution to the absolute value of the critical magnetic field is in general below 10%. Therefore, the values of ε determined by DB from their experimental data on the critical field may have a larger error. Another factor which should be pointed out is that, in the theory of this paper, we did not take into consideration the effect of impurities and stress, and we also assume that the energy gap is constant. Further studies have to be carried out on the effect of these factors to the $(\Delta t)^{3/2}$-order term of the critical field. However, the above qualitative results, at least, explain the following two points: (1) The fourth-order term of the field is very important in the contribution to the ε values; (2) The non-local effect of the film plays a decisive role in determining the variation of ε with d. It is because we not only included the 4th-order term of the field in the calculation, but also took into consideration the non-local effect of the film, that we find that $\varepsilon$ increases rapidly with decreasing d. If the local approximation of Gor'kov is used in the calculation, then the $\varepsilon$ obtained would not change with d even if the 4th-order term of the field is included.

In order to test formula (21), we compare it with the measurement results of Toxen (Ref. 4) on the critical magnetic field of In film. In (Ref. 8), Toxen plotted on a figure the experimental points of the ratio of the critical magnetic field, $H_c$, of In film and the critical magnetic field, $H_{cM}$, of a large sample of In at T = 0.9 $T_c$ and T = 0.95 $T_c$. For the calculation of $\frac{H_c}{H_{cM}}$ from the theoretical formula (21) of the film critical magnetic field, we use the material concerning the critical magnetic field of a large sample of In given by Muench (Ref. 12). According to the empirical formula given by Muench, the critical magnetic field of a large sample of In near $T_c$ can be expressed as

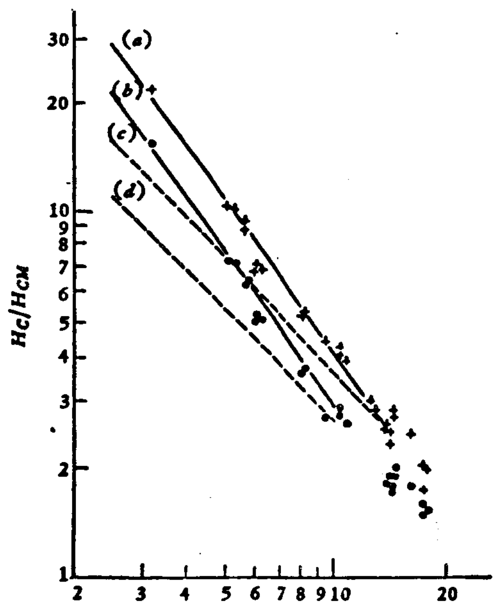
$$H_{cM} = 530\Delta t(1 - 0.395\Delta t). \tag{25}$$

From (21) and (25), we obtain

$$\frac{H_c}{H_{cM}} = \frac{2.36}{530}\frac{\hbar c}{e}\frac{(\Delta t)^{-1/2}}{\xi_0^{1/2} d^{3/2}[1-\mu(x)]^{1/2}} \times \times [1 + (\epsilon + 0.4)\Delta t]. \tag{26}$$

We calculated $\frac{H_c}{H_{cM}}$ in the range of the second-order phase transition according to (26). For good agreement of theory with experiment, $\xi_0$ ought to be selected as $\xi_0$ = 2.3 x $10^{-5}$ cm. This agrees with $\xi_0$ = 2.6 x $10^{-5}$ cm required in Toxen's own work.

In Figure 3, we have drawn two theoretical curves of $\frac{H_c}{H_{cM}} \sim$ d at /999 $\Delta t$ = 0.10 and $\Delta t$ = 0.05 (theoretical curves are drawn only for the case of the second-order phase transition region) and the experimental points read from Toxen's figure. For comparison, the results of the GL theory (2) are




Half Thickness d ($10^{-6}$ cm)

Figure 3

The Variation of $\frac{H_c}{H_{cM}}$ Values with d for In Film and the Experimental Points of Toxen

| | | |
|---|---|---|
| (a) | Theoretical curve of this paper | $\Delta t = 0.05$; |
| (b) | Theoretical curve of this paper | $\Delta t = 0.10$ |
| (c) | Theoretical curve of GL | $\Delta t = 0.05$; |
| (d) | Theoretical curve of GL | $\Delta t = 0.10$; |
| "+" | Experimental points of Toxen | $\Delta t = 0.05$; |
| "." | Experimental points of Toxen | $\Delta t = 0.10$ |

also drawn on the figure, using $\delta_0 = \frac{\delta_L(0)}{\sqrt{2\Delta t}}$ and taking $\delta_L(0) = 3.5 \times 10^{-6}$ cm (Ref. 8) at $\Delta t = 0.10$ and $\Delta t = 0.05$. The agreement of our theoretical values with the experimental values of Toxen is very satisfactory.

## APPENDIX

### Expansions of the Functions F(η) and G(η)

The definitions of functions $F(\eta)$ and $G(\eta)$ are as follows:

$$F(\eta) = \sum_{n=0}^{\infty} \frac{1}{2n+1}\left[1 - (2n+1)\eta \,\mathrm{tg}^{-1} \frac{1}{(2n+1)\eta}\right], \qquad \text{(A1)}$$

$$G(\eta)=\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\,\mathrm{tg}^{-1}\frac{1}{(2n+1)\eta}. \tag{A2}$$

To obtain their expansions at large $\eta$, we can find their Taylor expansion in $\frac{1}{\eta}$. We shall omit this very simple calculation. Here we shall only discuss the expansions of $F(\eta)$ and $G(\eta)$ for small $\eta$.

The expansions of $F(\eta)$ and $G(\eta)$ at small $\eta$ can be obtained by various methods. We will introduce one of them.

First, let us take $F(\eta)$. Let $\lambda = \frac{\pi}{\eta}$, then

$$\begin{aligned}F(\eta)\equiv F(\lambda)&=\sum_{n=0}^{\infty}\frac{\pi}{(2n+1)\pi}\left[1-\frac{(2n+1)\pi}{\lambda}\,\mathrm{tg}^{-1}\frac{\lambda}{(2n+1)\pi}\right]\\ &=\lim_{n\to\infty}\frac{i}{4}\oint_{\Gamma_n}\frac{\mathrm{tg}\,\frac{z}{2}}{z}\left(1-\frac{z}{\lambda}\,\mathrm{tg}^{-1}\frac{\lambda}{z}\right)dz.\end{aligned} \tag{A3}$$

where the single-valued branch of the multivalued function $\mathrm{tg}^{-1}\frac{\lambda}{z}$ is determined as follows, Make a branch cut on the z-plane from $i\lambda$ to $-i\lambda$, and set $\arg\left(\frac{z+i\lambda}{z-i\lambda}\right)=0$ when $z=+\infty$; $\{\Gamma n\}$ is the rectangular closed path as indicated in Figure 4.

For the single-valued branch of $\mathrm{tg}^{-1}\frac{\lambda}{z}$ defined above, it is very easy to prove that when $|z|$ is large

$$\left(1-\frac{z}{\lambda}\,\mathrm{tg}^{-1}\frac{\lambda}{z}\right)\sim\frac{1}{3}\left(\frac{\lambda}{z}\right)^2+0\left(\frac{\lambda}{z}\right)^4.$$

Moreover, it is not difficult to see that on the closed path $\{\Gamma n\}$, $\mathrm{tg}\,\frac{z}{2}$ is bounded. Therefore, as $n\to\infty$, the contribution from the integration over BC, CD and DA approaches zero, and only the integral along AB is left. Therefore, (A.3) can be converted to

$$F(\lambda)=\frac{i}{4}\int_{i\infty}^{-i\infty}\frac{\mathrm{tg}\,\frac{z}{2}}{z}\left(1-\frac{z}{\lambda}\,\mathrm{tg}^{-1}\frac{\lambda}{z}\right)dz, \tag{A.4}$$

/1000

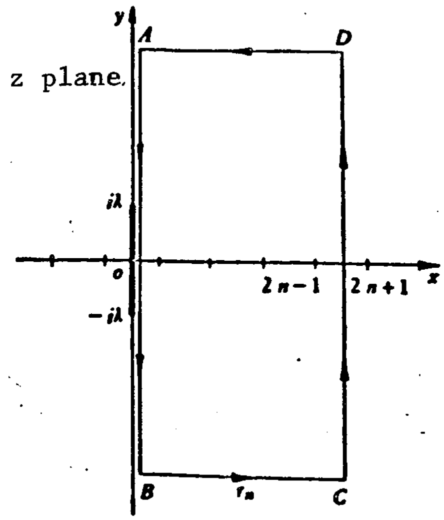



Figure 4

Closed Path $\Gamma n$

The heavy black line connecting $i\lambda$ and $-i\lambda$ in the figure indicates the branch cut.

where the integration path is along the imaginary axis on the right-hand side of the cut. Nothing the characteristics of the single-valued branch of $tg^{-1}\frac{\lambda}{z}$ defined above along the imaginary axis, we can see that

$$F(\lambda)=\frac{1}{2}\int_0^{\lambda}\frac{\operatorname{th}\frac{y}{2}}{y}\,dy-\frac{1}{2\lambda}\int_0^{\lambda}\operatorname{th}\frac{y}{2}\operatorname{th}^{-1}\frac{y}{\lambda}\,dy-\frac{1}{2\lambda}\int_{\lambda}^{\infty}\operatorname{th}\frac{y}{2}\left(\operatorname{th}^{-1}\frac{\lambda}{y}-\frac{\lambda}{y}\right)dy. \tag{A.5}$$

Each integral in (A.5) is separately calculated below. The first term in (A.5) is

$$I_1=\frac{1}{2}\int_0^{\lambda}\frac{\operatorname{th}\frac{y}{2}}{y}\,dy=\frac{1}{2}\ln\lambda\operatorname{th}\frac{\lambda}{2}-\frac{1}{4}\int_0^{\infty}\frac{\ln y}{\operatorname{ch}^2\frac{y}{2}}\,dy+\frac{1}{2}\int_{\lambda}^{\infty}\ln y\left(\operatorname{th}\frac{y}{2}\right)'dy,$$

Using the expansion

$$\operatorname{th}\frac{y}{2}=1+2\sum_{n=1}^{\infty}(-)^n e^{-ny}\qquad(y>0), \tag{A.6}$$

by carrying out integration term by term (term by term integration is allowed here), we obtain

$$I_1 = \frac{1}{2}\ln\frac{\lambda}{\pi} + \frac{1}{2}(C + \ln 2) + \sum_{n=1}^{\infty}(-)^n E_i(-n\lambda), \tag{A.7}$$

Discarding terms smaller than $0(e^{-\lambda})$, we have

$$I_1 = \frac{1}{2}\ln\frac{\lambda}{\pi} + \frac{1}{2}(C + \ln 2) + 0(e^{-\lambda}). \tag{A.8}$$

The second term in (A.5) is

$$\begin{aligned} I_2 = -\frac{1}{2\lambda}\int_0^\lambda \operatorname{th}\frac{y}{2}\operatorname{th}^{-1}\frac{y}{\lambda}\,dy = -\frac{1}{2\lambda}\int_0^\lambda \operatorname{th}^{-1}\frac{y}{\lambda}\,dy - \\ -\frac{1}{\lambda}\sum_{n=1}^{\infty}(-)^n\int_0^\lambda e^{-ny}\operatorname{th}^{-1}\frac{y}{\lambda}\,dy. \end{aligned} \tag{A.9}$$

Using the expansion

$$\operatorname{th}^{-1}\left(\frac{y}{\lambda}\right) = \sum_{s=0}^{\infty}\frac{1}{(2s+1)}\left(\frac{y}{\lambda}\right)^{2s+1} \quad (y < \lambda), \tag{A.10}$$

and integrating term by term (it is not hard to prove that this is allowed), (A.9) can be written as

$$\begin{aligned} I_2 = -\frac{1}{2}\sum_{s=0}^{\infty}\frac{1}{(2s+1)(2s+2)} - \sum_{n=1}^{\infty}(-)^n\sum_{s=0}^{\infty}\frac{1}{(2s+1)(n\lambda)^{2s+2}}\{(2s+1)! \\ - e^{-n\lambda}[(n\lambda)^{2s+1} + 2s(n\lambda)^{2s} + \cdots + (2s+1)!]\}. \end{aligned} \tag{A.11}$$

Neglecting $0\left(\frac{1}{\lambda^{2\nu}}\right)$ and small quantities less than $0(e^{-\lambda})$, we then have

$$\begin{aligned} I_2 = -\frac{1}{2}\sum_{s=0}^{\infty}\frac{1}{(2s+1)(2s+2)} + \sum_{s=0}^{N-1}\left(\frac{\pi}{\lambda}\right)^{2s+2}\frac{(2^{s+1}-1)}{(2s+1)(2s+2)}|B_{2s+2}| + \\ + 0\left(\frac{1}{\lambda^{2\nu+2}}, e^{-\lambda}\right), \end{aligned} \tag{A.12}$$

where $B_k$ are the Bernoulli numbers, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, .... For /1001 the third integral in (A.5),

$$I_3 = -\frac{1}{2\lambda}\int_\lambda^\infty \operatorname{th}\frac{y}{2}\left(\operatorname{th}^{-1}\frac{\lambda}{y} - \frac{\lambda}{y}\right)dy$$

it can be expressed in series form by the same method.

$$I_3 = -\frac{1}{2}\sum_{s=1}^{\infty}\frac{1}{2s(2s+1)} - \sum_{n=1}^{\infty}(-)^n e^{-n\lambda}\sum_{s=1}^{\infty}\frac{1}{2s(2s+1)}\left[1 - \frac{n\lambda}{(2s-1)} + \right.$$
$$\left. + \frac{(n\lambda)^2}{(2s-1)(2s-2)} - \cdots + \frac{(n\lambda)^{2s}}{(2s-1)!}\right] \tag{A.13}$$

$$= -\frac{1}{2}\sum_{s=1}^{\infty}\frac{1}{2s(2s+1)} + 0(e^{-\lambda}). \tag{A.14}$$

Summing up the above calculations, we obtain

$$F(\lambda) = I_1 + I_2 + I_3 = \frac{1}{2}\ln\frac{\lambda}{\pi} + \frac{1}{2}(C + \ln 2 - 1) +$$
$$+ \sum_{s=0}^{N-1}\left(\frac{\pi}{\lambda}\right)^{2s+2}\frac{(2^{2s+1}-1)|B_{2s+2}|}{(2s+1)(2s+2)} + 0\left(\frac{1}{\lambda^{2N+2}}, e^{-\lambda}\right), \tag{A.15}$$

In writing this expression, the equality $\sum_{k=1}^{\infty}\frac{1}{k(k+1)} = 1$ is used. Therefore, when $\eta < 1$:

$$F(\eta) = \frac{1}{2}\ln\frac{1}{\eta} + \frac{1}{2}(C + \ln 2 - 1) + \sum_{s=0}^{N-1}\frac{(2^{2s+1}-1)|B_{2s+2}|}{(2s+1)(2s+2)}\eta^{2s+2} +$$
$$+ 0(\eta^{2N+2}, e^{-\frac{\pi}{\eta}}). \tag{A.16}$$

This is the desired expansion.

By a completely similar method, we can obtain the expansion for $G(\eta)$ when $\eta < 1$:

$$G(\eta) = \frac{\pi^3}{16} - \frac{\eta}{2}\ln\frac{1}{\eta} - \frac{\eta}{2}(C + \ln 2 + 1) + \sum_{s=1}^{N-1}\frac{(2^{2s-1}-1)|B_{2s}|}{2s(2s+1)}\eta^{2s+1} +$$
$$+ 0(\eta^{2N+3}, e^{-\frac{\pi}{\eta}}). \tag{A.17}$$

Writing out the first several terms of (A.15) and (A.16), we have

$$F(\eta) = \frac{1}{2}\ln\frac{1}{\eta} + \frac{1}{2}(C + \ln 2 - 1) + \frac{1}{12}\eta^2 + \frac{7}{360}\eta^4 + 0(\eta^6, e^{-\frac{\pi}{\eta}}), \tag{A.18}$$

$$G(\eta) = \frac{\pi^3}{16} - \frac{\eta}{2}\ln\frac{1}{\eta} - \frac{\eta}{2}(C + \ln 2 + 1) + \frac{1}{36}\eta^3 + 0(\eta^5, e^{-\frac{\pi}{\eta}}). \tag{A.19}$$

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